JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 29, 273–293 (1970)

Singular Perturbations of Nonlinear Boundary Value Problems with Turning Points

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1. INTRODUCTION

This paper is concerned with the asymptotic behavior as $\epsilon \to 0^+$ of solutions $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ to nonlinear boundary value problems of the form

$$\begin{cases} u'' = f(t, u, v) \\ u(0) = u(1) = 0 \end{cases} \qquad (0 < t < 1), \tag{1.1}$$

$$\begin{cases} \epsilon v'' + g(t, u, u')\, v' - c(t, u, u')\, v = 0 \\ v(0) = v_0, \qquad v(1) = v_1 \end{cases} \qquad (0 < t < 1). \tag{1.2}$$

We assume that $0 \leqslant v_0 < v_1$ and $c(t, u, u') \geqslant 0$.

We are particularly concerned with problems in which there is exactly one interior turning point for equation (1.2). That is, for each $\epsilon > 0$ there is a unique point $\alpha \in (0, 1)$ such that $g(\alpha, u(\alpha), u'(\alpha)) = 0$, and $g(t, u(t), u'(t))$ changes sign in a neighborhood of $t = \alpha$. In general α depends on ϵ, and is not known a priori. This behavior occurs, for example, in the cases

$$f(t, u, v) = \pm v, \qquad g(t, u, u') = u'. \tag{1.3}$$

* Research supported by the Mathematics Research Center, U.S. Army, Madison, Wis., under Contract No. DA-31-124-ARO-D-462, by the Office of Naval Research under Contract No. N-00014-67-A-0128-004, and by the Atomic Energy Commission under Contract No. W-7405-ENG-36.

† Research supported by the Office of Naval Research under Contract No. N-00014-67-A-0128-004.

These problems may be considered as one-dimensional analogs of the steady-state Navier-Stokes equations in the form

$$\begin{cases} \Delta\psi = -\omega & \text{in } G, \\ \Delta\omega + R(\psi_x\omega_y - \psi_y\omega_x) = 0 & \text{in } G, \\ \psi, \omega \text{ prescribed on } \partial G, \end{cases} \tag{1.4}$$

where $R = 1/\epsilon$ is the Reynolds number. Problem (1.4) has been studied numerically as $R \to +\infty$ by Greenspan [10]. With his choice of boundary conditions the nonlinear partial differential equation always has an interior singular point ("stagnation point"), and the usual asymptotic analysis does not apply (see [14]). The asymptotic behavior of solutions to the Navier-Stokes equations has been studied by Batchelor [1–3] and others [4, 5, 8, 13, 18]. These authors, however, make substantial use of physical arguments as well as mathematical ones. In an effort to gain insight into such problems, we have therefore turned to the one-dimensional models (1.1)–(1.2).

There is an extensive literature on singular perturbation and turning-point problems for ordinary differential equations. A comprehensive bibliography is given in Wasow [21]. Specific examples of problems of the type we consider have been treated by Wasow [19, 20] and Cochran [6]. Macki [15] and Harris [11] have treated similar nonlinear first-order systems in which one equation is reduced in order as $\epsilon \to 0^+$.

In Section 2 we collect some preliminary results. Section 3 is devoted to Problems (1.3) and their generalizations. Problems with turning points at the ends of the interval are considered in Section 4. In Section 5 we study problems for which $c(t, u, u') \geqslant c_0 > 0$. In Section 6 we collect some remarks on further applications of the methods developed in this paper.

2. Preliminary Results

Before we can discuss the asymptotic behavior as $\epsilon \to 0^+$, we must establish two basic facts:

(a) For each $\epsilon > 0$ there exist solutions to equations (1.1)–(1.2).

(b) There are "limit pairs" $U(t)$, $V(t)$ of the family $\{u(t, \epsilon), v(t, \epsilon)\}$, and these limiting functions satisfy the "reduced equations" (in some sense).

These facts are not deep. The first follows from a fixed-point argument, and the second follows from a theorem of Friedrichs [9]. We will first give a complete discussion of these ideas.

Throughout the remainder of this paper we will assume that:

(H.1) $0 \leqslant v_0 < v_1$.

(H.2) $c(t, u, u') \geqslant 0$.

(H.3) $f(t, u, v)$, $g(t, u, u')$, and $c(t, u, u')$ are continuous in all variables.

(H.4)[1] There exists a continuous function $f_0(t, v)$ such that

$$|f(t, u, v)| \leqslant f_0(t, v)$$

for $t \in [0, 1]$ and $v \in [0, v_1]$.

THEOREM 1. *For each fixed $\epsilon > 0$, there exist solutions $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ in $C^2[0, 1]$ to equations* (1.1)–(1.2).

Proof. We note the following a priori estimates on solutions $u(t)$ and $v(t)$. The maximum principle [17] applied to equation (1.2) implies that

$$0 \leqslant v(t, \epsilon) \leqslant v_1 . \tag{2.1}$$

From (H.4) it follows that $|f(t, u, v)|$ is bounded, and hence from equation (1.1) we see that there is a constant M_1 such that $|u(t)|$, $|u'(t)|$, $|u''(t)| \leqslant M_1$. Finally, these estimates applied to equation (1.2) yield the existence of a constant M_2 such that $|v'(t)|$, $|v''(t)| \leqslant M_2$.

Let K be the set of function pairs $(u(t), v(t))$ in $C^2[0, 1] \times C^2[0, 1]$ such that:

(a) $u(0) = u(1) = 0$, $v(0) = v_0$, $v(1) = v_1$.

(b) $|u(t)| \leqslant M_1$, $|u'(t)| \leqslant M_1$, $|u''(t)| \leqslant M_1$.

(c) $0 \leqslant v(t) \leqslant v_1$, $|v'(t)| \leqslant M_2$, $|v''(t)| \leqslant M_2$.

Note that $\bar{K}$ is convex and compact in $C^1[0, 1] \times C^1[0, 1]$ by the Arzela-Ascoli Theorem [7, p. 266]. Let T be the operator defined on $C[0, 1] \times C[0, 1]$ by $T((u, v)) = (U, V)$, where

$$\begin{cases} U'' = f(t, u, v) \\ U(0) = U(1) = 0 \end{cases} \qquad (0 < t < 1)$$

$$\begin{cases} \epsilon V'' + g(t, u, u')\, V' - c(t, u, u')\, V = 0 \\ V(0) = v_0, \qquad V(1) = v_1 . \end{cases} \qquad (0 < t < 1)$$

The operator T is well-defined because of the maximum principle, and the a priori estimates mentioned earlier imply that $T : \bar{K} \to \bar{K}$. Hence we apply

[1] Many of the results of this paper are easily carried over to the case where (H. 4) is replaced by $\partial f/\partial u \geqslant \tau > -\pi^2$.

the Schauder fixed-point theorem [7, p. 456] to obtain a fixed pair (u, v) that satisfy equations (1.1)–(1.2).

We now turn to the consideration of "limit pairs." Assume that we have a sequence $\epsilon_n \to 0^+$ and a function $V(t)$ such that

$$\lim_{n\to\infty} v(t, \epsilon_n) = V(t)$$

pointwise almost everywhere (a.e.) on $[0, 1]$ (this then implies that $V(t) \in L^\infty[0, 1]$). Since $\{u(t, \epsilon_n)\}$, $\{u'(t, \epsilon_n)\}$, and $\{u''(t, \epsilon_n)\}$ are all uniformly bounded, there is a subsequence $\epsilon_{n(k)}$ of ϵ_n and a function $U(t) \in C^1[0, 1]$ such that $u(t, \epsilon_{n(k)})$ converges to $U(t)$ in the topology of $C^1[0, 1]$. Moreover, using the Green's function to obtain the integral equation satisfied by $u(t, \epsilon)$, and then letting $\epsilon = \epsilon_{n(k)} \to 0$, we find that

$$\begin{cases} U'' = f(t, U, V) \\ U(0) = U(1) = 0 \end{cases} \qquad (\text{a.e.}[0, 1]).$$

Motivated by these remarks, we define S_0 to be the set of all $V(t) \in L^1[0, 1]$ such that there exists a sequence $\epsilon_n \to 0$ with $\lim_{n\to\infty} v(t, \epsilon_n) = V(t)$ pointwise a.e. on $[0, 1]$. Then we have:

THEOREM 2. (a) *S_0 is not empty.* (b) *Let $g(t, u, u') \in C^1$, and let $(U(t), V(t))$ be a limit pair as described above. Assume that there is an interval $(a, b) \subset (0, 1)$ such that*

$$g(t, U(t), U'(t)) \neq 0 \qquad \text{for} \quad t \in (a, b).$$

Then $V(t) \in C^1(a, b)$, and

$$g(t, U, U')\, V' - c(t, U, U')\, V = 0 \qquad (a < t < b). \tag{2.2}$$

Proof. It follows from the maximum principle that $v(t, \epsilon)$ has no interior maximum and at most one interior minimum. Hence, using equation (2.1), we see that $\{v(t, \epsilon) \mid \epsilon > 0\}$ is of uniformly bounded total variation on $[0, 1]$. Thus statement (a) follows from the Helly selection theorem [16, p. 222], and in fact we can choose the sequence ϵ_n so that $\{v(t, \epsilon_n)\}$ converges to $V(t)$ for *all* $t \in [0, 1]$.

To prove statement (b), we define the differential operator

$$E\phi(t) = G(t)\,\phi'(t) - C(t)\,\phi(t) \qquad (a < t < b)$$

where $G(t) = g(t, U(t), U'(t))$ and $C(t) = c(t, U(t), U'(t))$. Because $U(t)$ and $U'(t)$ both satisfy a uniform Lipschitz condition on $[0, 1]$, it follows that $G(t)$ is absolutely continuous. A straightforward calculation then shows that $V(t)$ is a "weak" solution of

$$EV(t) = 0 \qquad (a < t < b). \tag{2.3}$$

Applying a theorem of Friedrichs,[2] we find that $V(t)$ is also a "strong" solution of equation (2.3). That is, if $R' = (a', b')$ is an arbitrary subinterval of (a, b) with $a < a' < b' < b$, then there exists a sequence $v_n(t) \in C_0^1(a, b)$ such that

$$\lim_{n\to\infty} \| v_n - v \|_{L^2(R')} = \lim_{n\to\infty} \| G v_n' - C v_n \|_{L^2(R')} = 0 \tag{2.4}$$

Since $| G(t)| \geqslant M_1 > 0$ on R', it follows that

$$\| v_n' \|_{L^2(R')} \leqslant M_2$$

for some constant M_2 independent of n. An elementary argument now shows that $V(t)$ is absolutely continuous on R'. Using the integral relations derived from equation (2.4), we find that the reduced equation (2.2) is satisfied on R'. Since R' is arbitrary, this completes the proof of Theorem 2.

3. Problems With Interior Turning Points

The problems we consider in this section are motivated by the special cases $f(t, u, v) = \pm v$, $g(t, u, u') = u'$, and $c(t, u, u') \equiv 0$. In these cases, $u'(t, \epsilon)$ has exactly one zero in $[0, 1]$, and this occurs at some point $\alpha(\epsilon) \in (0, 1)$. Furthermore, $u'(t, \epsilon)$ changes sign at $t = \alpha(\epsilon)$, so that equation (1.2) has exactly one (unknown) turning point at $\alpha(\epsilon)$.

The following observation is fundamental to much of our discussion: let $V(t) = V(t, \epsilon)$ satisfy the linear equation

$$\begin{cases} \epsilon V''(t) + G(t)\, V'(t) = 0 \\ V(0) = v_0, \ V(1) = v_1 \end{cases} \qquad (0 < t < 1),$$

where $G(t)$ is a known function. Then $V(t)$ is a monotone increasing function, which is given explicitly by

$$V(t, \epsilon) = v_0 + (v_1 - v_0)\Big[\int_0^t G_0(\tau, \epsilon)\, d\tau\Big]\Big[\int_0^1 G_0(\tau, \epsilon)\, d\tau\Big]^{-1}, \tag{3.1}$$

where

$$G_0(t, \epsilon) = \exp\Big[-\frac{1}{\epsilon}\int_0^t G(\tau)\, d\tau\Big]. \tag{3.2}$$

[2] We remark that the Friedrichs theorem [9, p. 135] requires $G \in C^1(a, b)$. However, it is clear in [9], and in the proof of this result given by Hormander [12], that the C^1 requirement is needed only to permit integration by parts. Thus the absolute continuity of G is sufficient to make the theorem applicable.

Another fundamental fact is the following refinement of Theorems 1 and 2 for these cases:

LEMMA 1. *Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to equations* (1.1)–(1.2) *with $c(t, u, u') \equiv 0$. Then we have:*

(a) *$v(t, \epsilon)$ is a strictly monotone increasing function for $0 < t < 1$. Assume that $f(t, u(t, \epsilon), v(t, \epsilon)) \neq 0$ for $0 < t < 1$. If $f(t, u, v) \leqslant 0$, then $u(t, \epsilon)$ is strictly concave, with exactly one maximum at $\alpha(\epsilon) \in (0, 1)$. If $f(t, u, v) \geqslant 0$, then $u(t, \epsilon)$ is strictly convex, with exactly one minimum at $\alpha(\epsilon) \in (0, 1)$. In either case $u''(\alpha(\epsilon), \epsilon) \neq 0$.*

(b) *There exist a sequence $\epsilon_n \to 0$, a constant $\alpha \in [0, 1]$, a function $U(t)$ with $U'(t)$ absolutely continuous, and a monotone nondecreasing function $V(t) \in L^1[0, 1]$ such that:*

(i) *$\{u(t, \epsilon_n)\}$ converges uniformly to $U(t)$,*

(ii) *$\{u'(t, \epsilon_n)\}$ converges uniformly to $U'(t)$,*

(iii) *$\{v(t, \epsilon_n)\}$ converges pointwise to $V(t)$,*

(iv) *$U(t)$ and $V(t)$ satisfy the differential equation*

$$\begin{cases} U'' = f(t, U, V) \\ U(0) = U(1) = 0 \end{cases} \quad \text{(a.e.}[0, 1]), \tag{3.3}$$

(v) *$\{\alpha(\epsilon_n)\}$ converges to α.*

Proof. The remarks concerning $u(t, \epsilon)$ and $v(t, \epsilon)$ follow immediately from the maximum principle. After extracting several subsequences, statements (i) and (ii) follow from the Arzela-Ascoli theorem, (iii) follows from the Helly selection theorem, and (v) follows from the Bolzano-Weierstrass theorem. Statement (iv) follows from the remarks preceding Theorem 2.

THEOREM 3. *Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to*

$$\begin{cases} u''(t) = v(t) & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ \epsilon v''(t) + u'(t)\, v'(t) = 0 & (0 < t < 1) \\ v(0) = v_0, \; v(1) = v_1 & \end{cases}$$

Let $d = v_0/v_1$ and $\alpha = (1 - \sqrt{d})/(1 - d)$. Then

$$\lim_{\epsilon \to 0} v(t, \epsilon) = \begin{cases} v_0 & 0 \leqslant t < \alpha \\ v_1 & \alpha < t \leqslant 1 \end{cases} \tag{3.4}$$

Proof. Let $U(t)$, $V(t)$, and α be the limits whose existence is guaranteed by Lemma 1. Since

$$-u(\alpha(\epsilon_n), \epsilon_n) = \| u(\cdot, \epsilon_n)\|_\infty \to \| U \|_\infty ,$$

we see that

$$U(\alpha) = -\| U \|_\infty . \tag{3.5}$$

Condition (ii) of Lemma 1 implies that

$$U'(\alpha) = 0. \tag{3.6}$$

Moreover, using the Green's function for equation (3.3) together with equation (3.6), we have

$$U(\alpha) = -\| U \|_\infty = -\int_0^\alpha tV(t)\, dt. \tag{3.7}$$

Assume that $U(t) \not\equiv 0$. Equation (3.5) and the fact that $U(0) = U(1) = 0$ imply that

$$0 < \alpha < 1.$$

We claim that the converse of Equation (3.6) holds in this case; i.e.,

$$U'(t_0) = 0 \qquad \text{implies} \qquad t_0 = \alpha. \tag{3.8}$$

Suppose that this is not the case. Then there exists a point $t_0 \neq \alpha$ such that $U'(t_0) = 0$. For the moment, assume that $t_0 \in [0, \alpha)$. Since $u'(t, \epsilon_n)$ is monotone increasing, $U'(t)$ is monotone nondecreasing. Thus,

$$U'(t) = 0 \qquad \text{for} \qquad t_0 \leqslant t \leqslant \alpha.$$

But then

$$V(t) = U''(t) = 0 \qquad (\text{a.e. } t_0 \leqslant t \leqslant \alpha).$$

Since $V(t)$ is monotone nondecreasing, we therefore have

$$V(t) = 0 \qquad (\text{a.e. } 0 \leqslant t \leqslant \alpha).$$

Using equation (3.7), we find that $\| U \|_\infty = 0$, which is a contradiction, and this proves statement (3.8) if $t_0 \in [0, \alpha)$. The case in which $t_0 \in (\alpha, 1]$ leads to a similar contradiction, and this completes the proof of statement (3.8).

In order to show that equation (3.4) is satisfied, we make use of the maximum principle and basic comparison functions. Still assuming that $U(t) \not\equiv 0$, let $t \in (0, \alpha)$, and set $\bar{t} = \frac{1}{2}(t + \alpha)$. We then have $U'(\tau) \leqslant U'(\bar{t}) < 0$ for $\tau \in [0, \bar{t}]$, so that for $n \geqslant N$,

$$u'(\tau, \epsilon_n) \leqslant M < 0 \qquad (0 \leqslant \tau \leqslant \bar{t}),$$

and the constant M is independent of n. Define $\phi(\tau, \epsilon)$ by

$$\phi(\tau, \epsilon) = [v(\tau, \epsilon) - v_0] \exp\left[\frac{M}{\epsilon}(\tau - \bar{t})\right]. \qquad (3.9)$$

Then $\phi(\tau) = \phi(\tau, \epsilon)$ satisfies the differential equation

$$\begin{cases} \epsilon\phi'' + (u' - 2M)\phi' + \dfrac{M}{\epsilon}(M - u')\phi = 0 & (0 < \tau < \bar{t}) \\ \phi(0) = 0, \quad \phi(\bar{t}) = v(\bar{t}) - v_0 . \end{cases}$$

Since $(M/\epsilon_n)(M - u') \leqslant 0$ for $0 \leqslant \tau \leqslant \bar{t}$, the maximum principle implies that

$$0 \leqslant \phi(\tau) \leqslant (v(\bar{t}) - v_0) \leqslant (v_1 - v_0).$$

But $0 < t < \bar{t}$ and $M < 0$, so by letting $\epsilon = \epsilon_n \to 0$ in equation (3.9) we see that $V(t) = v_0$ for $0 \leqslant t < \alpha$. To show that $V(t) = v_1$ for $\alpha < t \leqslant 1$, we use a similar argument with the comparison function

$$\phi(\tau, \epsilon) = [v_1 - v(\tau, \epsilon)] \exp\left[\frac{M}{\epsilon}(\tau - \bar{t})\right]$$

where $\bar{t} = \frac{1}{2}(t + \alpha)$ and $u'(\tau, \epsilon_n) \geqslant M > 0$ for $\bar{t} \leqslant \tau \leqslant 1$ and $n \geqslant N$.

If $v_0 > 0$, we have $V(t) \geqslant v_0 > 0$, so that $U(t) \not\equiv 0$. To complete the proof of the theorem in this case, we therefore need only determine the value of α and show that we can dispense with the selection of a sequence $\epsilon_n \to 0$. Using equation (3.3), we find that

$$U(t) = \begin{cases} \dfrac{1}{2} v_0 t \left[t - \alpha + \dfrac{2U(\alpha)}{\alpha v_0}\right] & 0 \leqslant t \leqslant \alpha \\ \dfrac{1}{2} v_1(t - 1)\left[t - \alpha + \dfrac{2U(\alpha)}{(\alpha - 1) v_1}\right] & \alpha \leqslant t \leqslant 1. \end{cases}$$

Since $U(t) \in C^1[0, 1]$ and $U'(\alpha) = 0$, by equating the two representations of $U'(\alpha)$, we derive the additional conditions

$$\alpha^2 v_0 = -2U(\alpha) = (\alpha - 1)^2 v_1 .$$

Solving this equation for the root $\alpha \in (0, 1)$, we find that

$$\alpha = (1 - \sqrt{d})/(1 - d), \qquad d = \frac{v_0}{v_1} .$$

Since the functions $U(t)$ and $V(t)$ are uniquely determined (except for $V(\alpha)$), we see that all convergent sequences must have the same limiting values.

Thus, the entire sequences of both functions converge, and this proves the theorem for the case $v_0 > 0$.

The proof of the theorem in the case $v_0 = 0$ proceeds by contradiction. For, assume that $U(t) \not\equiv 0$. The above argument then shows that $V(t) = 0$ for $0 \leqslant t < \alpha$, and by equation (3.7) we would then have $U(t) \equiv 0$. This therefore proves that $U(t) \equiv 0$, and hence $V(t) = 0$ for $0 \leqslant t < 1$. The uniqueness of the limit functions then enables us to complete the proof of the theorem.

The method of proof used in this theorem leads to the following generalizations. In both theorems we assume that $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ are solutions in $C^2[0, 1]$ to equations (1.1)–(1.2).

THEOREM 4. *Assume that*:

(a) *If* $v > 0$ *then* $f(t, u, v) > 0$.

(b) $f(t, u, 0) = 0$.

(c) *If* $u' \leqslant 0$, *then* $g(t, u, u') \leqslant 0$.

(d) *If* $t \in (0, 1)$, $u \leqslant 0$, *and* $g(t, u, u') = 0$, *then* $u' = 0$.

(e) $v_0 = 0$.

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = 0$ *for* $0 \leqslant t < 1$.

THEOREM 5. *Assume that*:

(a) $f(t, u, v)$ *is independent of* u (*write* $f(t, u, v) = f_0(t, v)$).

(b) *If* $v > 0$, *then* $f_0(t, v) > 0$.

(c) $f_0(t, 0) = 0$.

(d) $(u')\, g(t, u, u') \geqslant 0$.

(e) *If* $t \in (0, 1)$, $u \leqslant 0$, *and* $g(t, u, u') = 0$, *then* $u' = 0$.

(f) $c(t, u, u') \equiv 0$.

(g) $v_0 > 0$.

Let $V(t) \in S_0$. *Then there is an* $\alpha \in [0, 1]$ *such that*

$$V(t) = \begin{cases} v_0 & 0 \leqslant t < \alpha \\ v_1 & \alpha < t \leqslant 1. \end{cases}$$

Moreover, α *satisfies the equation*

$$\int_0^\alpha t f_0(t, v_0)\, dt = \int_\alpha^1 (1 - t)\, f_0(t, v_1)\, dt. \tag{3.10}$$

If equation (3.10) *has a unique solution* $\alpha \in [0, 1]$, *then* $V(t)$ *is unique, and we have*

$$\lim_{\epsilon \to 0} v(t, \epsilon) = \begin{cases} v_0 & 0 \leqslant t < \alpha \\ v_1 & \alpha < t \leqslant 1. \end{cases}$$

In the preceding example with $f(t, u, v) = v$, we have seen that, for $v_0 > 0$, the limit function $V(t)$ *retains both* boundary conditions. In the next case, $f(t, u, v) = -v$, the limit function *loses both* boundary conditions:

THEOREM 6. *Let* $u(t) = u(t, \epsilon)$ *and* $v(t) = v(t, \epsilon)$ *be solutions in* $C^2[0, 1]$ *to*

$$\begin{cases} u''(t) = -v(t) & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ \epsilon v''(t) + u'(t)\, v'(t) = 0 & (0 < t < 1) \\ v(0) = v_0, \quad v(1) = v_1. & \end{cases}$$

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = \frac{1}{2}(v_0 + v_1)$ for $0 < t < 1$.

The proof of this theorem proceeds with a sequence of lemmas. We first note that there is a unique $\alpha(\epsilon) \in (0, 1)$ such that $u'(\alpha(\epsilon), \epsilon) = 0$. If we let ϵ_n, $U(t)$, and $V(t)$ be as in Lemma 1, we then have:

LEMMA 2. $U(t) \not\equiv 0$

Proof. This is trivial if $v_0 > 0$, since then $V(t) \geqslant v_0 > 0$. Thus we consider the case $v_0 = 0$. Using equations (3.1)–(3.2), we have the integral representation

$$v(t, \epsilon) = v_1 \left[\int_0^t G_0(\tau, \epsilon)\, d\tau\right]\left[\int_0^1 G_0(\tau, \epsilon)\, d\tau\right]^{-1}, \tag{3.11}$$

where $G_0(\tau, \epsilon) = \exp[-(1/\epsilon)\, u(\tau, \epsilon)]$. Let $F(t) = u(t) - u(1 - t)$, so that $F(\frac{1}{2}) = F(1) = 0$ and $F''(t) = v(1 - t) - v(t)$. Since $v(t)$ is monotone increasing, $F''(t) \leqslant 0$ for $\frac{1}{2} \leqslant t \leqslant 1$. Thus

$$u(t) \geqslant u(1 - t) \qquad (\tfrac{1}{2} \leqslant t \leqslant 1).$$

Inserting this bound in equation (3.11) leads to the inequality

$$v(\tfrac{1}{2}, \epsilon) \geqslant \tfrac{1}{2} v_1 .$$

Thus, $V(\frac{1}{2}) \geqslant \frac{1}{2} v_1$, and so $U(t) \not\equiv 0$.

Define $w(t, \epsilon)$ by

$$w(t, \epsilon) = \frac{v(1 - t, \epsilon) - v(t, \epsilon)}{v_1 - v_0}.$$

From the integral representation (3.1)–(3.2), we see that

$$w(t, \epsilon) = \left[\int_t^{1-t} G_0(\tau, \epsilon)\, d\tau\right]\left[\int_0^1 G_0(\tau, \epsilon)\, d\tau\right]^{-1}.$$

LEMMA 3. *There is a constant $M > 0$ such that, for $0 < \epsilon \leqslant 1$ we have*

$$0 < \left[\int_0^1 G_0(\tau, \epsilon)\, d\tau\right]^{-1} \leqslant \frac{M}{\epsilon}.$$

Proof. Using the fact that $v(t, \epsilon) \leqslant v_1$, and the Green's function representation for $u(t, \epsilon)$, we find that

$$u(t, \epsilon) \leqslant \tfrac{1}{2}v_1(t - t^2).$$

Thus,

$$G_0(t, \epsilon) \geqslant \exp\left[-\frac{v_1}{2\epsilon}(t - t^2)\right],$$

and so

$$\begin{aligned}\int_0^1 G_0(\tau, \epsilon)\, d\tau &\geqslant 2\int_0^{\frac{1}{2}} \exp\left[-\frac{v_1}{2\epsilon}(\tau - \tau^2)\right] d\tau \\ &\geqslant 2\int_0^{\frac{1}{2}} \exp\left[-\frac{v_1}{2\epsilon}\tau\right] d\tau \\ &\geqslant \frac{\epsilon}{M}\end{aligned}$$

for

$$M = \frac{v_1}{4}\left[1 - \exp\left(-\frac{v_1}{4}\right)\right]^{-1}.$$

LEMMA 4. (a) $\lim_{n\to\infty} w(t, \epsilon_n) = 0$ for $0 < t < 1$.

(b) $V(t) = \frac{1}{2}(v_0 + v_1)$ for $0 < t < 1$.

Proof. Since $w(t, \epsilon) = -w(1 - t, \epsilon)$, it suffices to prove statement (a) for $t \in (0, \frac{1}{2}]$. It follows from the concavity of $u(t, \epsilon)$ that $U(t) = 0$ if and only if $t = 0$ or $t = 1$. Let $t \in (0, \frac{1}{2}]$ be fixed. Since $U(\tau) \neq 0$ for $\tau \in [t, 1 - t]$, there is a constant $M_1 = M_1(t)$ such that

$$U(\tau) \geqslant M_1 > 0 \qquad (t \leqslant \tau \leqslant 1 - t).$$

Since $\{u(t, \epsilon_n)\}$ converges uniformly to $U(t)$, there exists a constant $M_2 = M_2(t)$ and an integer N such that for $n \geqslant N$ we have

$$u(\tau, \epsilon_n) \geqslant M_2 > 0 \qquad (t \leqslant \tau \leqslant 1 - t).$$

Thus, for $n \geqslant N$,

$$0 \leqslant w(t, \epsilon_n) \leqslant \frac{M(1 - 2t)}{\epsilon_n} \exp\left(-\frac{1}{\epsilon_n} M_2\right).$$

This proves statement (a), which in turn implies that $V(t) = V(1 - t)$ for $0 < t < 1$. Since $V(t)$ is monotone nondecreasing, we see that $V(t) =$ constant for $0 < t < 1$. Denote this constant by $\bar{v}$, and notice that

$$U(t) = \tfrac{1}{2}\bar{v}(t - t^2).$$

Since $u(0, \epsilon) = u(1, \epsilon)$, the representation (3.1)–(3.2) implies that $v'(0, \epsilon) = v'(1, \epsilon)$. Thus

$$v_1 u'(1, \epsilon) - v_0 u'(0, \epsilon) = -\int_0^1 v^2(t, \epsilon)\, dt. \tag{3.12}$$

Letting $\epsilon = \epsilon_n \to 0$ in equation (3.12), we find that $\bar{v}$ satisfies the equation

$$\tfrac{1}{2}\bar{v}(2\bar{v} - v_0 - v_1) = 0.$$

Since $U(t) \not\equiv 0$, we have $\bar{v} > 0$, so that $\bar{v} = \frac{1}{2}(v_0 + v_1)$. The remainder of the proof of Theorem 6 follows in the same fashion as the proof of Theorem 3.

Remark. Since $\| U \|_\infty = U(\alpha)$ and $U(t) = \frac{1}{4}(v_0 + v_1)(t - t^2)$, we see that $\lim_{\epsilon \to 0} \alpha(\epsilon) = \frac{1}{2}$.

We state the following generalization of Theorem 6. The proof differs only in details, and hence is omitted.

THEOREM 7. *Let* $u(t) = u(t, \epsilon)$ *and* $v(t) = v(t, \epsilon)$ *be solutions in* $C^2[0, 1]$ *to equations* (1.1)–(1.2) *with* $c(t, u, u') \equiv 0$. *Assume that*:

(a) If $v > 0$ then $f(t, u, v) < 0$.

(b) $f(t, u, 0) = 0$.

(c) $(u')\, g(t, u, u') \geqslant 0$.

(d) If $t \in (0, 1)$, $u \geqslant 0$, and $g(t, u, u') = 0$, then $u' = 0$.

Let $V(t) \in S_0$. *Then there is a constant* $\bar{v} \in [v_0, v_1]$ *such that* $V(t) = \bar{v}$ *for* $0 < t < 1$. *If, in addition to the above, we have* $g(t, u, u') \in C^1$ *and*

$$\lim_{n \to \infty} \epsilon_n (v'(1, \epsilon_n) - v'(0, \epsilon_n)) = v^*, \tag{3.13}$$

then $\bar{v}$ *satisfies*

$$\begin{aligned} v^* &+ v_1 g(1, 0, U'(1)) - v_0 g(0, 0, U'(0)) \\ &= \bar{v} \int_0^1 \left[\frac{\partial g}{\partial t} + U' \frac{\partial g}{\partial u} + f(t, U, \bar{v}) \frac{\partial g}{\partial u'} \right] dt. \end{aligned} \tag{3.14}$$

The partial derivatives of g *in equation* (3.14) *are evaluated at* $(t, U(t), U'(t))$, *and* $U(t)$ *satisfies*

$$\begin{cases} U''(t) = f(t, U(t), \bar{v}) & (0 < t < 1) \\ U(0) = U(1) = 0. \end{cases}$$

Remark. If equation (3.13) holds for all sequences $\epsilon_n \to 0+$, and if $\bar{v}$ is unique, then $\lim_{\epsilon\to 0} v(t, \epsilon) = \bar{v}$ for $0 < t < 1$.

The calculation of v^* in equation (3.13) is not always an easy matter. In the case of Theorem 6 we had $v'(1, \epsilon) = v'(0, \epsilon)$, so that the computation of the limit $v^* = 0$ was immediate. We now discuss a nontrivial example for which v^* can be found, but we do not attempt to treat the most general problem which can be handled with these techniques.

THEOREM 8. *Let $k \geqslant 1$ be an integer, and let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to*

$$\begin{cases} u'' = -v & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ \epsilon v'' + (u')^{2k+1} v' = 0 & (0 < t < 1) \\ v(0) = v_0, \quad v(1) = v_1 . & \end{cases}$$

Let $V(t) \in S_0$. Then there is a constant $\bar{v}$ such that $V(t) = \bar{v}$ for $0 < t < 1$, and either

$$\bar{v} = 0 \qquad or \qquad \bar{v} = \tfrac{1}{2}(v_0 + v_1).$$

If $v_0 > 0$, we therefore have

$$\lim_{\epsilon\to 0} v(t, \epsilon) = \tfrac{1}{2}(v_0 + v_1) \qquad (0 < t < 1).$$

Proof. The existence of $\bar{v}$ follows from Theorem 7, and a straightforward calculation shows that equation (3.14) leads to

$$v^* - (\tfrac{1}{2}\bar{v})^{2k+1} (v_0 + v_1) = -(\tfrac{1}{2})^{2k} (\bar{v})^{2k+2}. \tag{3.15}$$

We will now show that

$$\bar{v} > 0 \qquad \text{implies that} \qquad v^* = 0. \tag{3.16}$$

The proof of the theorem will then follow in the usual fashion, with the use of Theorem 7 and equation (3.15).

Equations (3.1)–(3.2) yield the representation

$$v(t, \epsilon) = v_0 + (v_1 - v_0)\Big[\int_0^t G_0(\tau, \epsilon)\, d\tau\Big]\Big[\int_0^1 G_0(\tau, \epsilon)\, d\tau\Big]^{-1} \tag{3.17}$$

where $G(t, \epsilon) = \int_0^t [u'(\tau, \epsilon)]^{2k+1}\, d\tau$ and $G_0(t, \epsilon) = \exp[(-1/\epsilon)\, G(t, \epsilon)]$. Observe that

$$G''(t, \epsilon) = -(2k + 1)\, v(t, \epsilon)[u'(t, \epsilon)]^{2k}$$

so that $G(t, \epsilon)$ is concave. Using integration by parts, it is easy to show that

$$G(1, \epsilon) = -\sum_{j=1}^{k} C_{j-1} I_j ,$$

where

$$\begin{cases} I_j = \displaystyle\int_0^1 v' v^{j-1} u^{j+1} (u')^{2(k-j)}\, dt \\ C_j = \left(\dfrac{j+1}{j+2}\right) 2^{j+1} \displaystyle\prod_{n=0}^{j} \left(\dfrac{k-n}{n+1}\right). \end{cases}$$

Hence $G(1, \epsilon) < 0$, and so from equation (3.17) we see that $v'(1, \epsilon) > v'(0, \epsilon)$. Thus, for $k \geqslant 1$, the limit v^* cannot be calculated trivially, as it can for $k = 0$.

To verify statement (3.16), we will show that

$$G(1, \epsilon) = O(\epsilon^2) \qquad (\text{as } \epsilon \to 0+) \tag{3.18}$$

(for the sake of simplicity, we have deleted the subscript on ϵ_n). Using the concavity of $G(t, \epsilon)$, we see that there is a unique $\beta = \beta(\epsilon) \in (\alpha(\epsilon), 1)$ such that

$$G(t) \begin{cases} > 0 & 0 < t < \beta \\ = 0 & t = 0, \beta \\ < 0 & \beta < t \leqslant 1. \end{cases}$$

Since $u(t, \epsilon)$, $u'(t, \epsilon)$, and $v(t, \epsilon)$ are uniformly bounded, we have $| G(1, \epsilon)| \leqslant MJ(\epsilon)$, where

$$J = J(\epsilon) = \int_0^1 v'(t, \epsilon)\, u^2(t, \epsilon)\, dt$$

and M is a constant independent of ϵ. Let $J = \sum_{j=1}^{4} J_j$, where

$$J_j = \int_{a_j}^{b_j} v'(t, \epsilon)\, u^2(t, \epsilon)\, dt$$

and the intervals of integration are

$$(0, \sqrt{\epsilon}),\ (\sqrt{\epsilon}, \beta(\epsilon) - \sqrt{\epsilon}),\ (\beta(\epsilon) - \sqrt{\epsilon}, \beta(\epsilon)), \qquad \text{and} \qquad (\beta(\epsilon), 1).$$

Using the limiting form of $U(t)$, we see that

$$\lim_{\epsilon \to 0} G(1, \epsilon) = 0.$$

We are assuming that $\bar{v} > 0$, and since $U(t) = \frac{1}{2}\bar{v}(t - t^2) \not\equiv 0$, we have

$$\lim_{\epsilon \to 0} \beta(\epsilon) = 1.$$

Thus, the intervals of integration are well defined for ϵ sufficiently small.

Let $M > 0$ be a generic constant. By the same method used in the proof of Lemma 3, we see that

$$0 < v'(t, \epsilon) \leqslant \frac{M}{\epsilon} \exp\left[-\frac{1}{\epsilon} G(t, \epsilon)\right]. \tag{3.19}$$

Since $U'(0) = \frac{1}{2}\bar{v} > 0$, for ϵ small enough we have

$$G(t, \epsilon) \geqslant Mt \qquad (0 \leqslant t \leqslant \sqrt{\epsilon}).$$

Using the bound $0 \leqslant u(t, \epsilon) \leqslant Mt$, we find that

$$0 < J_1 \leqslant M\epsilon^2.$$

For the integral J_2, by using the concavity of $G(t, \epsilon)$ and the fact that $U(t) \not\equiv 0$, it is easy to show that

$$G(t, \epsilon) \geqslant M\sqrt{\epsilon} \qquad (\sqrt{\epsilon} \leqslant t \leqslant \beta(\epsilon) - \sqrt{\epsilon}).$$

Thus, for ϵ sufficiently small,

$$0 < J_2 \leqslant \frac{M}{\epsilon} \exp\left(-\frac{M}{\sqrt{\epsilon}}\right).$$

The technique used for J_1, together with the estimate $u(t, \epsilon) \leqslant M(1 - t)$, can be used to show that

$$0 < J_3 \leqslant M[(1 - \beta)^2 + \epsilon(1 - \beta) + \epsilon^2].$$

Finally, we note that J_4 can be bounded in the following way:

$$\begin{aligned}
J_4 &= \int_\beta^1 u^2(t)\, v'(t)\, dt \\
&= M\left[\int_\beta^1 u^2(t)\, G_0(t, \epsilon)\, dt\right]\left[\int_0^1 G_0(t, \epsilon)\, dt\right]^{-1} \\
&\leqslant M \max_{\beta \leqslant t \leqslant 1} u^2(t) \\
&\leqslant M(1 - \beta)^2.
\end{aligned}$$

From the mean-value theorem we have $G(1, \epsilon) = [1 - \beta(\epsilon)]\, G'(\tau, \epsilon)$, where $\beta(\epsilon) < \tau < 1$. Since $\beta(\epsilon) \to 1$ and $G'(1, \epsilon) \to [U'(1)]^{2k+1} < 0$, there exist constants $C_1, C_2 > 0$ so that, for ϵ small enough, we have

$$C_1 \mid G(1, \epsilon)| \leqslant | 1 - \beta(\epsilon)| \leqslant C_2 \mid G(1, \epsilon)|.$$

Combining all of the previous estimates, we have:

$$\begin{aligned}
| G(1, \epsilon)| &\leqslant MJ(\epsilon) \\
&\leqslant M\left[\epsilon^2 + \frac{1}{\epsilon} \exp\left(-\frac{M}{\sqrt{\epsilon}}\right) + (1 - \beta)^2 + \epsilon(1 - \beta)\right] \\
&\leqslant M\epsilon^2 + M \mid G(1, \epsilon)|(\epsilon + 1 - \beta) \\
&\leqslant M\epsilon^2.
\end{aligned}$$

This estimate shows that equation (3.18) is satisfied. Equations (3.17)–(3.19) then prove statement (3.16), which completes the proof of the theorem.

We observe that, in the case $v_0 = 0$ and $k \geqslant 1$, we have not eliminated the possibility of having a sequence $\{u(t, \epsilon_n)\}$ that converges uniformly to 0. However, in that case the *rate* of convergence can not be too rapid. Indeed, it can be shown that

$$\lim_{n\to\infty} \frac{1}{\epsilon_n} u(t, \epsilon_n) = +\infty \qquad (0 < t < 1).$$

4. Problems with Turning Points at the Ends of the Interval

The problems we consider in this section are motivated by the special cases $f(t, u, v) = \pm v$, $g(t, u, u') = u$, and $c(t, u, u') \equiv 0$. In these examples we no longer have an interior turning point, since $|u(t, \epsilon)| > 0$ for $t \in (0, 1)$. However, we have $u(t, \epsilon) = 0$ and $|u'(t, \epsilon)| > 0$ for $t = 0$ and $t = 1$, so there are "turning points" at each end of the interval. The asymptotic behavior is greatly simplified in this case: exactly one boundary condition is lost, and the one retained is determined by the sign of $u(t, \epsilon)$ for $t \in (0, 1)$. Because the proofs are essentially the same, we state the theorems in a general form and include the specific cases as examples. The functions $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ are solutions in $C^2[0, 1]$ to equations (1.1)–(1.2) with $c(t, u, u') \equiv 0$.

Theorem 9. *Assume that:*

(a) *If* $v > 0$, *then* $f(t, u, v) > 0$.

(b) *If* $u \leqslant 0$, *then* $g(t, u, u') \leqslant 0$.

(c) *If* $t \in (0, 1)$, $u \leqslant 0$, *and* $g(t, u, u') = 0$, then $u = 0$.

Then $\lim_{\epsilon\to 0} v(t, \epsilon) = v_0$ *for* $0 \leqslant t < 1$.

Example

$$\begin{cases} u'' = v \\ \epsilon v'' + uv' = 0. \end{cases}$$

Proof. Let $V(t) \in S_0$. If $U(t) \not\equiv 0$, then the general representation (3.1)–(3.2) can be used directly to show that

$$V(t) = v_0 \qquad (0 \leqslant t < 1).$$

The remainder of the proof follows as in the proof of Theorem 3.

THEOREM 10. *Assume that*:

(a) *If* $v > 0$, *then* $f(t, u, v) < 0$.

(b) *If* $u \geqslant 0$, *then* $g(t, u, u') \geqslant 0$.

(c) *If* $t \in (0, 1)$, $u \geqslant 0$, *and* $g(t, u, u') = 0$, *then* $u = 0$.

Then $\lim_{\epsilon\to 0} v(t, \epsilon) = v_1$ *for* $0 < t \leqslant 1$.

Example

$$\begin{cases} u'' = -v \\ \epsilon v'' + uv' = 0. \end{cases}$$

Proof. The proof is essentially the same as that of Theorem 9, and the details are again omitted. We remark that we always have $U(t) \not\equiv 0$, for

$$v'' = -\frac{1}{\epsilon} g(t, u, u')\, v' \leqslant 0,$$

and hence

$$v(t, \epsilon) \geqslant v_0 + (v_1 - v_0)t \qquad (0 \leqslant t \leqslant 1).$$

5. PROBLEMS WITH $c(t, u, u') \geqslant c_0 > 0$

In this section we consider problems for which the following additional hypotheses hold:

(H.5) $c(t, u, u') \geqslant c_0 > 0$,

(H.6) $g(t, u, u') \in C^1$.

These are included in a separate section because the asymptotic behavior is not determined by the nature of the turning points. Rather, it depends upon the fact that the reduced equations have no nontrivial solutions.

Recall that Theorem 2(b) states the following: if $V(t) \in S_0$ and $(a, b) \subset (0, 1)$ are such that $g(t, U(t), U'(t)) \neq 0$ for $t \in (a, b)$, then $V(t) \in C^1(a, b)$ and

$$G(t)\, V'(t) - C(t)\, V(t) = 0 \qquad (a < t < b),$$

where $G(t) = g(t, U(t), U'(t))$ and $C(t) = c(t, U(t), U'(t))$. Then for any $t_0 \in (a, b)$ we have

$$V(t) = V(t_0) \exp\left[\int_{t_0}^{t} \frac{C(\tau)}{G(\tau)}\, d\tau\right].$$

Assume that:

$$\begin{cases} \text{(a)}\ G(t) > 0 \quad \text{for} \quad t \in (a, b), \\ \text{(b)}\ \lim\limits_{t \to b^-} G(t) = 0, \\ \text{(c)}\ C(t) \geqslant c_0 > 0 \text{ in some interval } [b - \delta, b],\ \delta > 0.^3 \end{cases} \tag{5.1}$$

We can then conclude that $V(t) = 0$ for $t \in (a, b)$. For, suppose we fix $t_0 \in (a, b)$. It is easy to show that $G(t)$ satisfies a uniform Lipschitz condition on $[0, 1]$, and this fact, together with the assumptions in (5.1), implies that

$$\lim_{t \to b^-} \int_{t_0}^{t} \frac{C(\tau)}{G(\tau)}\, d\tau = +\infty.$$

Since $V(t)$ is bounded, we have $V(t_0) = 0$. A similar argument shows that $V(t) = 0$ for $t \in (a, b)$ if the following hold:

$$\begin{cases} \text{(a)}\ G(t) < 0 \quad \text{for} \quad t \in (a, b), \\ \text{(b)}\ \lim\limits_{t \to a^+} G(t) = 0, \\ \text{(c)}\ C(t) \geqslant c_0 > 0 \text{ in some interval } [a, a + \delta],\ \delta > 0. \end{cases}$$

If we have more precise information about the form of $f(t, u, v)$ and $g(t, u, u')$, we can make a stronger statement. Assume that we have:

$$\begin{cases} \text{(a)}\ f(t, u, v) = v f_1(t, u), \text{ and } |f_1(t, u)| \geqslant \tau > 0. \\ \text{(b)}\ g(t, u, u') = u' g_1(t, u), \text{ and if } t \in (0, 1) \text{ and } g_1(t, u) = 0, \text{ then } u = 0. \end{cases} \tag{5.2}$$

Also assume that:

(a) There exists a point $\alpha \in (0, 1)$ such that $U'(\alpha) = 0$.
(b) There exists an interval $I = (\alpha - \delta, \alpha)$ or $I = (\alpha, \alpha + \delta)$ with $\delta > 0$ such that $U'(t) \neq 0$ if $t \in I$.

We can then conclude that $V(t) = 0$ for $t \in I$. For, suppose $V(t) \not\equiv 0$ for $t \in I$. Then we must have $U(t) \not\equiv 0$, and hence by the convexity of $U(t)$ (or concavity, depending upon the sign of $f_1(t, u)$), we have $U(t) \neq 0$ for $t \in I$. Thus $G(t) \neq 0$ for $t \in I$. Suppose $I = (\alpha - \delta, \alpha)$, and let $t_0 \in I$. If $t \in (t_0, \alpha)$, from Theorem 2(b) we have

$$U'(t) = U'(t_0) \exp \left[\int_{t_0}^{t} F(\tau)\, d\tau \right], \tag{5.3}$$

[3] This condition is implied by (H. 5), but it is included here for the sake of completeness.

where $F(t) = f_1(t, U(t))\, g_1(t, U(t))\, V'(t)(C(t))^{-1}$. Now $U'(t) \in C[0, 1]$ and $F(t) \in C(I)$, so we let $t \to \alpha$ in equation (5.3) to find that $U'(t_0) = 0$. Since $t_0 \in I$ is arbitrary, we have $U'(t) = 0$ (and hence $V(t) = 0$) for $t \in I$. The proof for $I = (\alpha, \alpha + \delta)$ follows in the same way.

The next three theorems follow directly from the above remarks. The functions $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ are solutions in $C^2[0, 1]$ to equations (1.1)–(1.2), and the theorems give sufficient conditions for the following conclusion:

$$\lim_{\epsilon \to 0} v(t, \epsilon) = 0 \qquad \text{for} \quad 0 < t < 1. \tag{A}$$

THEOREM 11. *Assume that:*

(a) *If* $v \neq 0$, *then* $f(t, u, v) \neq 0$.

(b) $g(t, u, u') = 0$ *if and only if* $u = 0$.

Then (A) *holds.*

Examples

$$\begin{cases} u'' = \pm v. \\ \epsilon v'' + uv' - cv = 0. \end{cases}$$

THEOREM 12. *Assume that:*

(a) *If* $v > 0$, *then* $f(t, u, v) < 0$.

(b) $f(t, u, 0) = 0$.

(c) *If* $u' \geqslant 0$, *then* $g(t, u, u') \geqslant 0$.

(d) $g(t, u, 0) = 0$.

(e) *If* $t \in (0, 1)$, $u \geqslant 0$, *and* $g(t, u, u') = 0$, *then* $u' = 0$.

Then (A) *holds.*

Example

$$\begin{cases} u'' = -v \\ \epsilon v'' + u'v' - cv = 0. \end{cases}$$

THEOREM 13. *Assume that condition* (5.2) *is satisfied. Then* (A) *holds.*

Examples

$$\begin{cases} u'' = \pm v \\ \epsilon v'' + u'v' - cv = 0. \end{cases}$$

6. Other Problems

In this section we give two examples of problems not covered by previous theorems. They deal with situations in which $g(t, u(t), u'(t))$ is allowed to have a zero, but the function does not change sign in a neighborhood of this zero. The proofs follow directly with the use of techniques already developed and hence are omitted.

Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to equations (1.1)–(1.2) with $c(t, u, u') \equiv 0$.

THEOREM 14. *Assume that*:

(a) *If* $v \neq 0$, *then* $f(t, u, v) \neq 0$.

(b) $f(t, u, 0) = 0$.

(c) $g(t, u, u') \leqslant 0$.

(d) *If* $t \in (0, 1)$ *and* $g(t, u, u') = 0$, *then either* $u = 0$ *or* $u' = 0$.

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = v_0$ *for* $0 \leqslant t < 1$.

Examples

$$\begin{cases} u'' = \pm v \\ \epsilon v'' - (u')^{2k} v' = 0. \end{cases}$$

$$\begin{cases} u'' = \pm v \\ \epsilon v'' - (u)^{2k} v' = 0. \end{cases}$$

Remark. The conclusion of Theorem 14 remains valid for the general case $c(t, u, u') \geqslant 0$ if $v_0 = 0$. The method of proof follows that of Theorem 3.

THEOREM 15. *Assume that*:

(a) *If* $v \neq 0$, *then* $f(t, u, v) \neq 0$.

(b) $f(t, u, 0) = 0$.

(c) $g(t, u, u') \geqslant 0$.

(d) *If* $t \in (0, 1)$ *and* $g(t, u, u') = 0$, *then either* $u = 0$ *or* $u' = 0$.

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = v_1$ *for* $0 < t \leqslant 1$.

Examples

$$\begin{cases} u'' = \pm v \\ \epsilon v'' + (u')^{2k} v' = 0. \end{cases}$$

$$\begin{cases} u'' = \pm v \\ \epsilon v'' + (u)^{2k} v' = 0. \end{cases}$$

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